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Convection in a fluid with two phases

By F. H. BUSSE AND G. SCHUBERT
Department of Planetary and Space Science,
University of California,
Los Angeles, California 90024

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The gravitational instability of a horizontal fluid layer with a univariant phase transition is considered. It is found that the layer can be unstable even when the less dense phase lies above the dense phase and can be stable in the opposite case. Applications of the theory to convection with phase transitions in astrophysical and geophysical problems are briefly discussed.

1. Introduction

Convection in planetary interiors and atmospheres, and in stars, frequently involves phase transitions in the convecting medium. Often the latent heat liberated by a phase transition acts as the principal driving force for the convective motions. In other cases the co-existence of two phases may have a strongly stabilizing influence. Since phase transitions interact in a number of ways with convective motions, heuristic discussions (Vening Meinesz 1962; Knopoff 1964; Verhoogen 1965) of these interactions have sometimes led to controversial results. Thus it is desirable to have a mathematical analysis of the problem which takes into account the influences of density change and latent heat release on the dynamics as well as the dependence of the position of the phase boundary on the convective temperature and pressure fields. The present paper intends to provide a simple mathematical model which exhibits the characteristic features of convection in a fluid with phase transitions.

Idealized models have played an important role in the understanding of convective processes since Lord Rayleigh (1916) first gave a theoretical treatment of the problem of convection in a layer of fluid heated from below. Lord Rayleigh used the Boussinesq approximation and assumed a uniform temperature gradient and stress-free boundaries of infinite thermal conductivity. Models such as this will be modified in the present investigation by the addition of terms in the basic equations which describe the processes occurring at the interface between the two phases of the fluid. A δ-function representation of these terms permits a simple analytical treatment of the problem. Throughout the paper we restrict the analysis to linear equations, regarding the convective motions as small non-oscillatory disturbances of the basic state. In special cases it can be proved that growing oscillatory disturbances cannot exist. In general we shall assume the validity of the principle of exchange of stabilities without proof. In most cases the physically realized instability of the static state will set in as finite amplitude convection at values of the Rayleigh number below the critical value given by linear theory. Even in these cases, however, the linear equations provide a reasonable approximate description of the instability. We shall return to this point in the discussion at the end of the paper.

After deriving the basic equations in §2, the simplest case of convection involving a phase transition will be considered in §3. In this case the coefficient of thermal expansion is neglected and an adiabatic temperature gradient is assumed for the basic state. The static state is unstable only when the dense phase lies above the lighter one. In the case of two immiscible fluids the Rayleigh–Taylor instability cannot be prevented by viscous dissipation. However, when the boundary separates two phases of the same fluid, dissipation can play a stabilizing role. A case of particular interest is considered in §4 in which the effect of thermal expansion is still neglected but the static temperature gradient is no longer assumed to be adiabatic. Although the static state of a single-phase fluid is stable in this case, a two-phase fluid with the heavy phase below the light one may be unstable. The complementary case of a non-vanishing coefficient of expansion in the presence of an adiabatic temperature gradient is discussed in §5. Some properties of the general case are studied in §6. The fact that convective motions due to thermal expansion may or may not be influenced by a phase transition leads to the interesting phenomenon that the fluid layer may become unstable to convective modes of quite different scale at the same critical temperature gradient.

2. Basic equations

We consider a horizontal fluid layer of thickness d. The temperatures at the lower and upper boundaries are held at the constant values T_1 and T_2, respectively. The fluid consists of two phases which coexist at values of the pressure p and the temperature T satisfying the relation

$$f(p,T) = 0. \tag{2.1}$$

It is assumed that the fluid is homogeneous in the horizontal directions. Accordingly a static state of the fluid layer exists with a horizontal univariant phase boundary separating the two phases. The slope of the curve described by (2.1) is determined by the Clausius–Clapeyron relation

$$\left(\frac{dT}{dp}\right)_C = \frac{-\partial f/\partial p}{\partial f/\partial T} = \frac{T\Delta\rho}{q\rho_1\rho_2}, \tag{2.2}$$

where ρ_1 and ρ_2 are the densities of the two phases, $\Delta\rho$ is the density change in the transition from the less dense to the denser phase and q denotes the latent heat per unit mass released in this transition. We assume that the variation of density throughout the fluid including the change $\Delta\rho$ at the phase boundary is small compared with the mean density ρ_0. According to Le Chatellier's principle the denser phase corresponds to a higher pressure at constant temperature. Phase transitions are characterized in general by a positive function $(dT/dp)_C$. Excluding some rather exceptional cases we shall therefore assume that $q > 0$ in the following.

In order to separate the changes taking place in the phase transition from the changes taking place without transition the specific heat at constant pressure c_p and the expansion coefficient are written in the form

$$\left.\begin{aligned} c_p &= c+q(\partial f/\partial T)\,\delta(f), \\ -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_p &= \alpha+\frac{\Delta\rho}{\rho_0}\frac{\partial f}{\partial T}\delta(f), \end{aligned}\right\} \tag{2.3}$$

where $\delta(f)$ denotes the Dirac δ-function. The dependence of density on pressure is given by

$$-\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial p}\right)_T = \frac{\Delta\rho}{\rho_0}\frac{\partial f}{\partial p}\delta(f). \tag{2.4}$$

The function $f(p, T)$ has been chosen in such a way that it is positive for the light phase and negative for the dense phase. For simplicity we shall assume that c and α have the same constant value for each of the two phases. Similarly it will be assumed that the thermal diffusivity κ and the kinematic viscosity ν are constant throughout the fluid layer. The temperature dependence of the density will be taken into account in the gravitational body force term only. The dependence of density on pressure has been neglected entirely within each of the two phases. On the other hand, the pressure term in the energy equation will be retained in analogy to Jeffreys' (1930) formulation of the Boussinesq approximation for convection in compressible media. For details on the Boussinesq approximation we refer to Spiegel & Veronis (1960).

In order to obtain a dimensionless description of the problem we shall use d, d^2/κ, and q/c as scales for length, time and temperature, respectively. The equations of motion for the dimensionless velocity vector $\mathbf{u}$ and the energy equation for the dimensionless temperature Θ are

$$\left.\begin{aligned} &\frac{\kappa}{\nu}\frac{D}{Dt}\mathbf{u} = -\nabla\Pi-\frac{\rho}{\rho_0}\frac{gd^3}{\nu\kappa}\mathbf{k}+\nabla^2\mathbf{u}, \quad \nabla.\mathbf{u} = 0, \\ &\left\{1+\frac{\partial f}{\partial \Theta}\delta(f)\right\}\frac{D}{Dt}\Theta-\left\{\alpha T+\frac{cT\Delta\rho}{\rho_0 q}\frac{\partial f}{\partial\Theta}\delta(f)\right\}\frac{\nu\kappa}{qd^2}\frac{D\Pi}{Dt} = \nabla^2\Theta, \end{aligned}\right\} \tag{2.5}$$

where $\mathbf{k}$ is the unit vector in the direction opposite to the force of gravity and Π is the dimensionless pressure $pd^2/\nu\kappa\rho_0$. Viscous dissipation has been neglected in the energy equation. It is assumed that an instantaneous thermodynamic equilibrium takes place on the time scale d^2/κ. The dimensionless form of the Clausius–Clapeyron equation is

$$\left(\frac{d\Theta}{d\Pi}\right)_C = -\frac{\partial f/\partial\Pi}{\partial f/\partial\Theta} = \frac{\nu\kappa}{qd^2}\frac{\Delta\rho}{\rho_0}\Theta. \tag{2.6}$$

Equations (2.4) admit the static solution

$$\Theta = \Theta_0 \equiv \frac{T_1+(z-z_1)(T_2-T_1)}{q/c}; \quad \Pi = \Pi_0 \equiv -\int\frac{\rho g d^3}{\rho_0\nu\kappa}dz. \tag{2.7}$$

A Cartesian co-ordinate system has been introduced with the z co-ordinate in the direction of the unit vector $\mathbf{k}$ in such a way that

$$f(\Pi_0, \Theta_0) = 0 \quad \text{at} \quad z = 0. \tag{2.8}$$

The lower and upper boundaries are given accordingly by $z = z_1$ and $z = z_2$, respectively, with $z_2 - z_1 = 1$ and $z_2 z_1 < 0$.

To analyse the stability of the static solution we consider disturbances $\mathbf{u}, \theta, \pi$ of infinitesimal amplitude superimposed on the static solution. The equations for the perturbations are

$$\frac{\kappa}{\nu}\frac{\partial}{\partial t}\mathbf{u} = -\nabla\pi + \mathbf{k}\left\{\frac{\alpha g d^3 q}{c\nu\kappa}\theta + \frac{\Delta\rho g d^3}{\rho_0 \nu\kappa}\delta(f)\left(\frac{\partial f}{\partial\Theta_0}\theta + \frac{\partial f}{\partial\Pi_0}\pi\right)\right\} + \nabla^2\mathbf{u}, \tag{2.9}$$

$$\nabla . \mathbf{u} = 0, \tag{2.10}$$

$$\left(1 + \frac{\partial f}{\partial\Theta_0}\delta(f)\right)\frac{\partial\theta}{\partial t} - \left(\alpha T_0 + \frac{cT_0\Delta\rho}{q\rho_0}\frac{\partial f}{\partial\Theta_0}\delta(f)\right)\frac{\kappa\nu}{qd^2}\frac{\partial\pi}{\partial t}$$

$$= \mathbf{u}.\mathbf{k}\left(s_{12}\delta(z) - \frac{d\Theta_0}{dz} + \frac{\alpha T_0 \nu\kappa}{qd^2}\frac{d\Pi_0}{dz}\right) + \nabla^2\theta, \tag{2.11}$$

where

$$s_{12} = \pm 1 \quad \text{for} \quad \left(\frac{d\Theta_0}{d\Pi_0}\right)_0 - \left(\frac{d\Theta}{d\Pi}\right)_C \gtrless 0, \tag{2.12}$$

i.e. s_{12} is positive if heavy material lies above light material and negative if the light phase is on top. Here, and in the following, f is considered a function of Π_0, Θ_0 unless indicated otherwise. The subscript 0 at a bracket refers to $z = 0$.

The term $s_{12}\mathbf{u}.\mathbf{k}\delta(z)$, on the right-hand side of the energy equation, represents the contribution of the latent heat of transformation to the change in enthalpy of a fluid particle undergoing the phase transition. It arises in the following manner. From equation (2.5) we see that

$$s_{12}\delta(z) = -\frac{\partial f}{\partial\Theta_0}\delta(f)\left(\frac{d\Theta_0}{dz} - \frac{\nu\kappa\Theta_0\Delta\rho}{qd^2\rho_0}\frac{d\Pi_0}{dz}\right). \tag{2.13}$$

Using the relation $\delta((df/dz)_0 z) = |(df/dz)_0|^{-1}\delta(z)$ and equation (2.6) the above expression may be written

$$s_{12}\delta(z) = -\frac{\partial f}{\partial\Theta_0}\left|\left(\frac{df}{dz}\right)_0\right|^{-1}\delta(z)\left(\frac{d\Theta_0}{d\Pi_0} - \left(\frac{d\Theta}{d\Pi}\right)_C\right)\frac{d\Pi_0}{dz}.$$

We note from (2.1) and (2.6) that

$$\left(\frac{df}{dz}\right)_0 = \left(\frac{\partial f}{\partial\Theta_0}\right)_0\left\{-\left(\frac{d\Theta}{d\Pi}\right)_C + \left(\frac{d\Theta_0}{d\Pi_0}\right)_0\right\}\left(\frac{d\Pi_0}{dz}\right)_0,$$

which, together with the fact that $(\partial f/\partial\Theta_0)_0(d\Pi_0/dz)_0$ is a negative quantity, establishes relation (2.12).

According to the horizontal momentum balance π is of the order of $|\nabla\times\mathbf{u}|$ and it can therefore be neglected inside the wavy bracket of (2.9) where it is multiplied by the small quantity $\Delta\rho/\rho_0$, i.e. we neglect the force arising from the distortion of the interface due to the change in pressure π. We shall neglect the time derivative of π in (2.9) as well by assuming that the factor multiplying this term is sufficiently small.

With the aid of (2.13) and the definitions

$$R_\alpha = \frac{\alpha g d^3 q}{c\nu\kappa}, \quad R_\beta = -\frac{c}{q}\left\{T_2 - T_1 + \frac{\alpha T_0 g d}{c}\right\}, \qquad P = \frac{\Delta\rho}{\rho_0}\left|\left(\frac{d\Theta_0}{d\Pi_0}\right)_0 - \left(\frac{d\Theta}{d\Pi}\right)_C\right|^{-1}, \quad Q = P\frac{\nu\kappa\rho_0}{gd^3\Delta\rho}, \tag{2.14}$$

equations (2.9)–(2.11) take the simpler forms

$$(\kappa/\nu)\,\partial\mathbf{u}/\partial t = -\nabla\pi + \mathbf{k}(R_\alpha + P\delta(z))\,\theta + \nabla^2\mathbf{u}, \tag{2.15}$$

$$\nabla.\mathbf{u} = 0, \tag{2.16}$$

$$(1 + Q\delta(z))\,\partial\theta/\partial t = \mathbf{u}.\mathbf{k}(s_{12}\delta(z) + R_\beta) + \nabla^2\theta. \tag{2.17}$$

The parameter R_α is the Rayleigh number as in ordinary Bénard convection except that q/c replaces the temperature difference $T_1 - T_2$ between the boundaries. R_β represents the difference between the actual temperature gradient and the adiabatic lapse rate.

From the relation $f(\Pi_0 + \pi, \Theta_0 + \theta) = 0$ we obtain for the distortion of the interface between the two phases

$$\eta = -\left\{\frac{\dfrac{\partial f}{\partial\Theta_0}\theta + \dfrac{\partial f}{\partial\Pi_0}\pi}{\left(\dfrac{\partial f}{\partial\Theta_0}\right)_0\dfrac{d\Theta_0}{dz} + \left(\dfrac{\partial f}{\partial\Pi_0}\right)_0\dfrac{d\Pi_0}{dz}}\right\}_0. \tag{2.18}$$

Since the dependence on π can be neglected in comparison to the dependence on θ in accordance with earlier assumptions, expression (2.18) can be simplified to

$$\eta = \frac{s_{12}P\nu\kappa\rho_0\theta_{z=0}}{gd^3\Delta\rho}. \tag{2.19}$$

Thus depending on whether the actual ratio between temperature and pressure gradient is less or larger than the right-hand side of (2.2) the distortion of the interface has the same or the opposite sign as the temperature disturbance θ.

To eliminate the equation of continuity (2.16) we introduce in place of $\mathbf{u}$ the general representation of a solenoidal vector field

$$\mathbf{u} = \nabla\times(\nabla\times(\mathbf{k}v)) + \nabla\times(\mathbf{k}w). \tag{2.20}$$

The equation for w derived from (2.15) is

$$\frac{\kappa}{\nu}\frac{\partial}{\partial t}\Delta_2 w = \nabla^2\Delta_2 w, \tag{2.21}$$

where the operator Δ_2 is

$$\Delta_2 \equiv \nabla^2 - (\mathbf{k}.\nabla)^2.$$

Since (2.21) admits only decaying solutions for homogeneous boundary conditions, w can be neglected in the stability analysis. Two equations for v and θ can be derived from (2.15) and (2.17),

$$\left.\begin{aligned}\frac{\kappa}{\nu}\frac{\partial}{\partial t}\nabla^2 v &= \nabla^4 v-(P\delta(z)+R_\alpha)\,\theta,\\ (1+Q\delta(z))\,\partial\theta/\partial t &= \nabla^2\theta-(s_{12}\delta(z)+R_\beta)\,\Delta_2 v.\end{aligned}\right\} \tag{2.22}$$

These equations will be considered in the following sections together with two kinds of boundary conditions. In general we shall assume 'free' boundaries at which the normal component of the velocity and the viscous stress vanish,

$$v=\partial^2 v/\partial z^2=\theta=0 \quad \text{at} \quad z=z_1, z_2. \tag{2.23}$$

The results derived for 'free' boundaries differ usually only quantitatively from the results for rigid boundaries at which the total velocity vector vanishes,

$$v=\partial v/\partial z=\theta=0 \quad \text{at} \quad z=z_1, z_2. \tag{2.24}$$

For this reason 'rigid' boundaries will be considered only in qualitative discussions of the problem.

3. The case $R_\alpha \approx R_\beta \approx 0$

We consider the case when the temperature gradient is nearly equal to the adiabatic lapse rate, $R_\beta \approx 0$. The parameter R_α is the product of the magnitude of the adiabatic temperature gradient and the quantity $qd^3/T_0\,\nu\kappa$. Thus independently of the actual magnitude of the adiabatic lapse rate we can choose $R_\alpha \ll 1$ by considering $qd^3/T_0\nu\kappa \ll 1$. Consequently we first treat the simplest case $R_\alpha \approx 0$ and $R_\beta \approx 0$. Since the time dependence of v, θ can be assumed in the form $\exp(\sigma t)$, the equations (2.22) can be written

$$\left.\begin{aligned}\nabla^4 v-P\delta(z)\,\theta &= (\kappa/\nu)\,\sigma\nabla^2 v,\\ \nabla^2\theta-s_{12}\delta(z)\,\Delta_2 v &= \sigma\theta(1+Q\delta(z)).\end{aligned}\right\} \tag{3.1}$$

The operator on the left-hand side of this system of equations is self-adjoint if boundary conditions of the form (2.23), (2.24) are assumed. Consequently the growth rate σ of the unstable modes is real. To show this we multiply the first equation in (3.1) by $\Delta_2 v^*$, the second by $P\theta^*$ (the asterisk denotes the complex conjugate), average the equations and add to obtain

$$\begin{aligned}-\langle|\mathbf{k}\times\nabla\nabla^2 v|^2\rangle-P\langle|\nabla\theta|^2\rangle-P\langle\delta(z)\,(\theta\Delta_2 v^*+s_{12}\theta^*\Delta_2 v)\rangle\\ =\sigma[(\kappa/\nu)\,\langle|\mathbf{k}\times\nabla\nabla v|^2\rangle+P\langle|\theta|^2+Q|\theta_{z=0}|^2\rangle].\end{aligned} \tag{3.2}$$

Since P and Q are positive, all disturbances must decay in time if $s_{12}=-1$. Hence in the following, our attention will be restricted to the case $s_{12}=+1$. In this case only real values of σ are admissible and the stability of the static state will be determined by disturbances with $\sigma=0$. The general solution of (3.1) for disturbances of this kind with the boundary conditions (2.23) is

$$v=f(x,y)\,v_\gamma(z), \quad \theta=f(x,y)\,\theta_\gamma(z), \tag{3.3}$$

where $\gamma = 1$ for $z < 0$ and $\gamma = 2$ for $z > 0$. The function $f(x, y)$ satisfies

$$\Delta_2 f = -a^2 f,$$

and v_γ, θ_γ are given by

$$v_\gamma = A_\gamma \sinh a(z-z_\gamma) + B_\gamma(z-z_\gamma)\cosh a(z-z_\gamma),$$
$$\theta_\gamma = C_\gamma \sinh a(z-z_\gamma).$$

The coefficients $A_\gamma, B_\gamma, C_\gamma$ are determined by the following equations derived from (3.1):

$$\left.\begin{aligned}
&\sum_{\gamma=1}^{2} (A_\gamma \sinh a|z_\gamma| + B_\gamma |z_\gamma| \cosh a|z_\gamma|) = 0,\\
&\sum_{\gamma=1}^{2} (A_\gamma a \cosh a|z_\gamma| + B_\gamma(a|z_\gamma| \sinh a|z_\gamma| + \cosh a|z_\gamma|)) = 0,\\
&\sum_{\gamma=1}^{2} B_\gamma a \sinh a|z_\gamma| = 0,\\
&\sum_{\gamma=1}^{2} C_\gamma \sinh a|z_\gamma| = 0,\\
&B_1 a^2 \cosh az_1 - B_2 a^2 \cosh az_2 = -\tfrac{1}{2} P C_1 a \cosh az_1,\\
&C_1 a \cosh az_1 - C_2 a \cosh az_2 = a^2(A_1 \sinh az_1 + B_1 z_1 \cosh az_1).
\end{aligned}\right\} \quad (3.4)$$

The system (3.4) is solvable when the coefficient determinant of the unknowns $A_\gamma, B_\gamma, C_\gamma$ vanishes. This condition yields the characteristic equation

$$\begin{aligned}
0 = {} & 2a^2(\cosh az_1 + [\sinh a|z_1|/\sinh az_2]\cosh az_2)^3\\
& + P \sinh a|z_1| \{a(|z_1| + z_2[\sinh^2 az_1/\sinh^2 az_2])\\
& - \sinh a|z_1| (\cosh az_1 + [\sinh a|z_1|/\sinh az_2]\cosh az_2)\}. \quad (3.5)
\end{aligned}$$

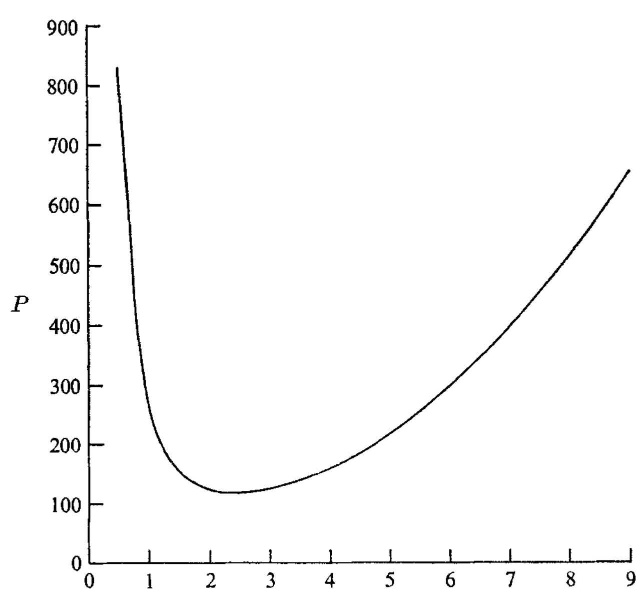

FIGURE 1. The curve of marginal stability, P *vs.* a, for the case $|z_1| = |z_2|$, $R_\alpha = R_\beta = 0$, $s_{12} = 1$.

It is of interest to evaluate (3.5) in the limits a tending to infinity and to zero,

$$a \to \infty, \quad P \to 8a^2, \tag{3.6}$$

$$a \to 0, \quad P \to 3/a^2|z_1 z_2|^3. \tag{3.7}$$

For $a \to \infty$, P becomes independent of the location of the boundaries. The lowest value of P is attained in the symmetric case $|z_1| = |z_2|$, for which the numerical evaluation of (3.5) yields the results shown in figure 1. The critical value of P at which the layer becomes unstable is 118·2 corresponding to $a = 2{\cdot}4$.

The physical interpretation of these results is that a static state in which the denser phase lies above the lighter phase can be stable if viscous and thermal dissipation are sufficiently strong to overcome the destabilizing force of gravity by keeping P below its critical value. This result contrasts with that in the case of two immiscible fluids where dissipation cannot prevent the Rayleigh–Taylor instability.

4. The case $R_\alpha \approx 0$, $R_\beta \neq 0$

When the assumption $R_\beta \approx 0$ is dropped the boundary-value problem is no longer self-adjoint. However, we shall still restrict our attention to disturbances with $\sigma = 0$ and consider the equations

$$\left.\begin{aligned} &\nabla^4 v - P\delta(z)\,\theta = 0, \\ &\nabla^2\theta - (R_\beta + s_{12}\delta(z))\,\Delta_2 v = 0. \end{aligned}\right\} \tag{4.1}$$

As in the preceding section the solution can be written in the form (3.3) with v_γ, θ_γ defined by

$$v_\gamma = A_\gamma \sinh a(z-z_\gamma) + B_\gamma(z-z_\gamma)\cosh a(z-z_\gamma),$$

$$\theta_\gamma = -\tfrac{1}{2}aR_\beta[(z-z_\gamma)A_\gamma \cosh a(z-z_\gamma) + \tfrac{1}{2}B_\gamma\{(z-z_\gamma)^2 \sinh a(z-z_\gamma) - (1/a)(z-z_\gamma)\cosh a(z-z_\gamma)\} + C_\gamma \sinh a(z-z_\gamma)].$$

The equations for the coefficients $A_\gamma, B_\gamma, C_\gamma$ lead to a characteristic equation in analogy to (3.5). In the symmetric case $|z_1| = |z_2| = \frac{1}{2}$, the equation is

$$(8a^3/R_\beta P)\cosh^3 \tfrac{1}{2}a - \frac{as_{12}}{R_\beta}\sinh \tfrac{1}{2}a\,(\sinh \tfrac{1}{2}a \cosh \tfrac{1}{2}a - \tfrac{1}{2}a) + \tfrac{1}{4}a^2 \sinh \tfrac{1}{2}a + \tfrac{3}{4}a\cosh \tfrac{1}{2}a - \tfrac{3}{2}\cosh^2 \tfrac{1}{2}a \sinh \tfrac{1}{2}a = 0. \tag{4.2}$$

In the limit $a \to \infty$ (4.2) yields

$$R_\beta = (16a^3/3P) - (2a/3)\,s_{12}, \tag{4.3}$$

while for $a \to 0$ we obtain

$$R_\beta = 480/Pa^2 - \tfrac{5}{2}s_{12}. \tag{4.4}$$

The characteristic equation (4.2) is of particular interest in the case $s_{12} = -1$, i.e. when the light phase lies above the heavy phase. Although this density stratification appears to be stable, it may in fact become unstable if a sufficiently large temperature difference is applied at the boundaries. The parameter R_β must exceed a positive value, depending on a, in order to overcome the stabilizing effect of the negative s_{12}. The minimum value of R_β required for instability is given by

$$R_{\beta m} = \min\left(\frac{3\cosh\frac{1}{2}a}{2a\sinh\frac{1}{2}a} - \tfrac{1}{4}a\,\frac{1}{\cosh\frac{1}{2}a\sinh\frac{1}{2}a - \frac{1}{2}a}\right)^{-1} = \tfrac{5}{2}, \tag{4.5}$$

corresponding to $a = 0$. The marginal stability curves R_β *vs.* a for various values of P are given in figure 2.

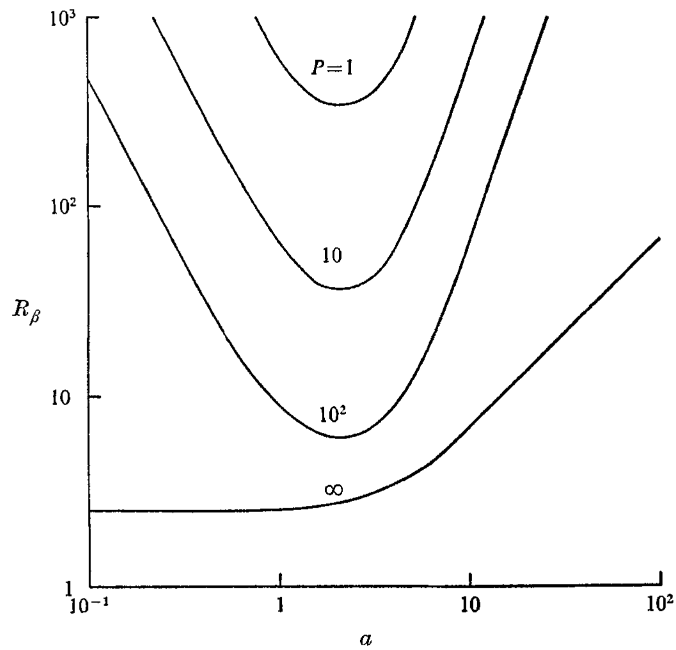

FIGURE 2. The marginal stability curves, R_β *vs.* a, for several values of P, for the case $|z_1| = |z_2|$, $R_\alpha = 0$, $s_{12} = -1$.

The instability is at first sight surprising since a flow conserving the thermodynamic state of the material would cause a stabilizing distortion of the interface between the phases. According to the energy equation, however, the thermodynamic state of the advected material is not conserved. Hence, depending on the advected temperature field, a distortion of the interface in the direction opposite to the flow becomes possible. This behaviour of the interface provides the key to understanding the instability mechanism. Where the flow is downward, for example, the phase boundary is distorted upward and a vertical column of fluid is heavier than an unperturbed fluid column. Thus the gravitational body force is downward, aiding the motion and producing the instability. Had the interface distorted downward, with the flow, the perturbed vertical column would have been relatively lighter and the resulting upward buoyancy force would have promoted stability. In his discussion of the inhibiting effect of phase transitions on convective motions Verhoogen (1965) does not include the destabilizing effect of the distortion of the interface.

Relation (4.2) also shows that a negative value of R_β has a stabilizing influence on the instability considered in the preceding section. However, this stabilizing influence becomes less significant as $a \to \infty$ as can be seen by rewriting (4.3) in the form

$$P = \frac{8a^2}{\frac{3}{2}(R_\beta/a) + s_{12}}.$$

5. The case $R_\beta \approx 0$, $R_\alpha \neq 0$

The analysis of the preceding section can be readily applied to the complementary case $R_\beta \approx 0$, $R_\alpha \neq 0$. Since the adjoint problem of (4.1) is given by

$$\left.\begin{aligned} \nabla^4 v^+ - (R_\beta + s_{12}\delta(z))\,\theta^+ &= 0, \\ \nabla^2\theta^+ - P\delta(z)\Delta_2 v^+ &= 0, \end{aligned}\right\} \tag{5.1}$$

with v^+, θ^+ satisfying the same boundary conditions as v, θ, it can be concluded that the dispersion relation (4.2) holds for the solution of the equations

$$\left.\begin{aligned} \nabla^4 v - (R_\alpha + P\delta(z))\,\theta &= 0, \\ \nabla^2\theta - s_{12}\delta(z)\,\Delta_2 v &= 0, \end{aligned}\right\} \tag{5.2}$$

if the parameters R_β, P and s_{12} in relation (4.2) are replaced by R_α, s_{12} and P, respectively. The asymptotic relations for the boundary of marginal stability follow from (4.3), (4.4):

$$\left.\begin{aligned} P &\approx (8a^2/s_{12}) - R_\alpha(3/2a) \quad \text{for} \quad a \to \infty, \\ P &\approx (192/a^2 s_{12}) - \tfrac{2}{5}R_\alpha \quad \text{for} \quad a \to 0. \end{aligned}\right\} \tag{5.3}$$

Since P and R_α are restricted to positive values in general, instability can occur only for $s_{12} = 1$ corresponding to the case when the dense phase is above the less dense one. For this reason a non-vanishing parameter R_α does not introduce new physical aspects into the problem other than a destabilizing influence on the instability considered in §3.

6. The general case

In the general case, $R_\alpha \neq 0$, $R_\beta \neq 0$, ordinary Rayleigh convection is possible for $R_\alpha R_\beta > 0$ when the phase transition is vanishing. This has the interesting consequence that two different modes of convection may simultaneously be possible. To discuss this phenomenon we consider solutions of (2.22) in the case $|z_1| = |z_2|$. Equations (2.22) have the stationary solutions with antisymmetric z dependence

$$\left.\begin{aligned} v_2 &= \sin 2\pi z f_2(x, y), \\ \theta_2 &= ((2\pi)^2 + a_2^2)^{-2} v_2, \end{aligned}\right\} \tag{6.1}$$

with $\Delta_2 f_2 = -a_2^2 f_2$. The motion described by (6.1) involves separate convection cells above and below the phase change interface. Since there is no motion through the phase boundary the parameters P and s_{12} do not influence this solution. The minimum value of $R_\alpha R_\beta$ for which a solution of the form (6.1) exists is $R_{c2} = 108\pi^4$

(Chandrasekhar 1961, pp. 35–36). Ordinarily solutions with a symmetric z dependence correspond to much lower critical Rayleigh numbers. However, this need not always be the case as can readily be seen in the limit $P \gg R_\alpha$. In this case the analysis of § 4 will be approximately valid even though R_α does not vanish. Since P is very large the symmetric mode will become unstable with a small value of the horizontal wave-number a when R_β just exceeds $\frac{5}{2}$. The anti-symmetric mode (6.1) with horizontal wave-number $a_2 = \pi\sqrt{2}$ will simultaneously become unstable at the value $R_\alpha \approx \frac{2}{5}R_{c2} = 216\pi^4/5$. It is interesting to speculate about the non-linear interaction of these two modes. In the present study, however, we shall not discuss this topic further.

When R_α and R_β are large compared to P and unity, respectively, or alternatively when $\Delta\rho/\rho \ll \alpha|T_2 - T_1|$ and $q \ll c|T_2 - T_1|$, the effect of the phase transition can be considered as a perturbation upon convection in a homogeneous fluid. Since the problem of convection in a homogeneous fluid is self-adjoint the influence of the perturbation is readily evaluated by multiplying the first and second of equations (2.22) by $R_\beta \Delta_2 v_1$ and $R_\alpha \theta_1$, respectively, and adding the averaged results. The functions v_1, θ_1 denote the solution of the stationary equations without the terms multiplied by P and s_{12} when $R_\alpha R_\beta$ has reached the critical value $R_{c1} = \frac{27}{4}\pi^4$ (Chandrasekhar 1961) of the Rayleigh number. The result

$$(R_\beta P + R_\alpha s_{12}) \langle \Delta_2 v_1 \theta_1 \delta(z) \rangle = (-R_\alpha R_\beta + R_{c1}) \langle \Delta_2 v_1 \theta_1 \rangle,$$

shows that the phase transition with negative s_{12} exerts a stabilizing or destabilizing influence depending on whether P is smaller or larger than R_α/R_β.

7. Concluding remarks

Linear stability analysis provides sufficient conditions for instability. Yet the fluid layer is not necessarily stable below the critical Rayleigh number established by the linear analysis. In the case of a single-phase fluid it has been found that instability sets in at subcritical Rayleigh numbers in the form of finite amplitude hexagonal convection when the material properties of the fluid depend on the temperature (see, for example, Busse 1967). A phase transition similarly favours instability in the form of subcritical hexagonal convection especially when the interface between the phases is not at the midpoint of the layer. Non-linear terms caused by the finite distortion of the interface also favour hexagonal convection. The terms can be derived in analogy to the linear term taken into account in equations (2.15)–(2.17).

The analysis described in the preceding sections pertains to a number of geophysical and astrophysical convection problems. Phase transitions with the less dense phase above the more dense one, as seem to occur in relatively deep layers of the earth's mantle, can contribute to instability by the mechanism discussed in § 4 as has been pointed out by Schubert, Turcotte & Oxburgh (1970). Recently Press (1969) has reported the results of a Monte Carlo procedure used to find density distributions for the upper mantle consistent with the geophysical data. These results suggest that a density inversion associated with a phase change may occur at a depth between 200 and 300 km. The analysis of § 3 indicates that a stationary state of convection is compatible with this situation. The phase change of water in clouds has a more complicated character owing to the dominant presence of air. An extension of the analysis of this paper may provide a quantitative model for cloud convection. Convection in the solar atmosphere is driven partly by the phase transition corresponding to the ionization of hydrogen. In this and other cases a simple model has the advantage that an analytical treatment of the non-linear effects seems feasible in analogy to the treatment of the case of a single-phase fluid.

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